SECTION TOPIC ▪
Integration by parts ▪

8.2 Integration by parts

The first new integration technique we present in this chapter is called **integration by parts.** This technique applies to a wide variety of functions and is particularly useful for integrands involving a *product* of algebraic and transcendental functions. For instance, integration by parts works well with integrals like $\int x \ln x\, dx$, $\int x^2e^x\, dx$, and $\int e^x \sin x\, dx$.

Integration by parts is based on the formula for the derivative of a product

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'$$

where both u and v are differentiable functions of x. If u' and v' are continuous, then we can integrate both sides of this equation to obtain

$$uv = \int uv'\, dx + \int vu'\, dx = \int u\, dv + \int v\, du$$

By rewriting this equation, we obtain the following theorem.

THEOREM 8.1 **INTEGRATION BY PARTS**

If u and v are functions of x and have continuous derivatives, then

$$\int u\, dv = uv - \int v\, du$$

This formula expresses the original integrand in terms of another integral. Depending on the choices for u and dv, it may be easier to evaluate the second integral than the original one. Since the choices of u and dv are critical in the integration by parts process, we suggest the following guidelines.

GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration formula. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a simpler function than u. Then dv will be the remaining factor(s) of the integrand.

Remark These are only suggested guidelines, and they should not be followed blindly or without some thought. Furthermore, it is usually best to consider assigning dv first.

EXAMPLE 1 Integration by parts

Evaluate $\int xe^x\,dx$.

Solution: To apply integration by parts, we want to write the integral in the form $\int u\,dv$. There are several ways to do this.

$$\int \underbrace{(x)}_{u}\underbrace{(e^x\,dx)}_{dv}, \qquad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv}, \qquad \int \underbrace{(1)}_{u}\underbrace{(xe^x\,dx)}_{dv}, \qquad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

Following our guidelines, we choose the first option, since e^x is the most complicated portion of the integrand that fits a basic integration formula. Thus, we have

$$dv = e^x\,dx \Rightarrow v = \int dv = \int e^x\,dx = e^x$$

$$u = x \Rightarrow du = dx$$

Now, by the integration by parts formula, we have

$$\int u\,dv = uv - \int v\,du$$

$$\int x\,e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Remark Note in Example 1 that it is not necessary to include a constant of integration when solving $v = \int e^x\,dx = e^x + C_1$. To illustrate this, we replace v by $v + C_1$ in the general formula to obtain

$$\int u\,dv = u(v + C_1) - \int (v + C_1)\,du = uv + C_1u - \int C_1\,du - \int v\,du$$

$$= uv + C_1u - C_1u - \int v\,du = uv - \int v\,du$$

EXAMPLE 2 Integration by parts

Evaluate $\int x^2 \ln x\,dx$.

Solution: In this case x^2 is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. Therefore, we let $dv = x^2\,dx$.

$$dv = x^2\,dx \Rightarrow v = \int x^2\,dx = \frac{x^3}{3}$$

$$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$$

Therefore, we have

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

One unusual application of integration by parts involves integrands consisting of a single factor, such as $\int \ln x\, dx$ or $\int \arcsin x\, dx$. In such cases, we let $dv = dx$, as illustrated in the next example.

EXAMPLE 3 An integrand with a single term

Evaluate

$$\int_0^1 \arcsin x\, dx$$

Solution: Letting $dv = dx$, we have

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \arcsin x \quad \Rightarrow \quad du = \frac{1}{\sqrt{1-x^2}}\, dx$$

Therefore, we have

$$\begin{aligned}\int \arcsin x\, dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx \\ &= x \arcsin x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\, dx \\ &= x \arcsin x + \sqrt{1-x^2} + C\end{aligned}$$

Now, using this antiderivative, we evaluate the definite integral as follows:

$$\begin{aligned}\int_0^1 \arcsin x\, dx &= \left[x \arcsin x + \sqrt{1-x^2}\right]_0^1 \\ &= \frac{\pi}{2} - 1\end{aligned}$$

The area represented by this definite integral is shown in Figure 8.1.

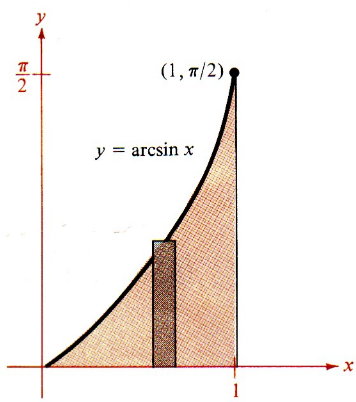

FIGURE 8.1

It may happen that an integral requires repeated application of the integration by parts formula. This is demonstrated in the next example.

EXAMPLE 4 Repeated application of integration by parts

Evaluate $\int x^2 \sin x\, dx$.

Solution: We may consider x^2 and $\sin x$ to be equally easy to integrate. However, the derivative of x^2 becomes simpler, whereas the derivative of $\sin x$ does not. Therefore, we let $u = x^2$ and write

$$dv = \sin x\, dx \Rightarrow v = \int \sin x\, dx = -\cos x$$

$$u = x^2 \Rightarrow du = 2x\, dx$$

and it follows that

$$\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx$$

Now, we apply integration by parts to the new integral. We let $u = 2x$ and write

$$dv = \cos x\, dx \Rightarrow v = \int \cos x\, dx = \sin x$$

$$u = 2x \Rightarrow du = 2\, dx$$

and it follows that

$$\int 2x \cos x\, dx = 2x \sin x - \int 2 \sin x\, dx = 2x \sin x + 2 \cos x + C$$

Combining these two results, we have

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

When making repeated application of integration by parts, you need to be careful not to interchange the substitutions in successive applications. For instance, in Example 4 our first substitution was $u = x^2$ and $dv = \sin x\, dx$. If, in the second application, we had switched the substitution to

$$dv = 2x\, dx \Rightarrow v = \int 2x\, dx = x^2$$

$$u = \cos x \Rightarrow du = -\sin x\, dx$$

we would have obtained

$$\begin{aligned}\int x^2 \sin x\, dx &= -x^2 \cos x + \int 2x \cos x\, dx \\ &= -x^2 \cos x + x^2 \cos x + \int x^2 \sin x\, dx \\ &= \int x^2 \sin x\, dx\end{aligned}$$

thus undoing the previous integration and returning to the *original* integral.

When making repeated applications of integration by parts, you should also watch for the appearance of a *constant multiple* of the original integral. This is illustrated in the next example.

EXAMPLE 5 Repeated application of integration by parts

Evaluate $\int e^x \cos 2x \, dx$.

Solution: Our guidelines fail to help with a choice of u and dv, so we arbitrarily choose $dv = e^x \, dx$ and $u = \cos 2x$. (You might try verifying that the choice of $dv = \cos 2x \, dx$ and $u = e^x$ works equally well.)

$$dv = e^x \, dx \quad \Rightarrow \quad v = \int e^x \, dx = e^x$$

$$u = \cos 2x \quad \Rightarrow \quad du = -2 \sin 2x \, dx$$

and it follows that

$$\int e^x \cos 2x \, dx = e^x \cos 2x + 2 \int e^x \sin 2x \, dx$$

Making the same type of substitutions for the next application of integration by parts, we have

$$dv = e^x \, dx \quad \Rightarrow \quad v = \int e^x \, dx = e^x$$

$$u = \sin 2x \quad \Rightarrow \quad du = 2 \cos 2x \, dx$$

and it follows that

$$\int e^x \sin 2x \, dx = e^x \sin 2x - 2 \int e^x \cos 2x \, dx$$

Therefore, we have

$$\int e^x \cos 2x \, dx = e^x \cos 2x + 2 \, e^x \sin 2x - 4 \int e^x \cos 2x \, dx$$

Now, since the right-hand integral is a constant multiple of the original integral, we add it to the left side of the equation to obtain

$$5 \int e^x \cos 2x \, dx = e^x \cos 2x + 2 \, e^x \sin 2x$$

$$\int e^x \cos 2x \, dx = \frac{1}{5} e^x \cos 2x + \frac{2}{5} e^x \sin 2x + C \qquad \text{Divide by 5}$$

The integral in the next example is an important one. In Section 8.4 we will see that it is used in finding the arc length of a parabolic segment. (See Example 8 in Section 8.4.)

EXAMPLE 6 Integration by parts

Evaluate $\int \sec^3 x\, dx$.

Solution: The most complicated portion of the integrand that can be easily integrated is $\sec^2 x$. Letting $dv = \sec^2 x\, dx$ and $u = \sec x$, we have

$$dv = \sec^2 x\, dx \Longrightarrow v = \int \sec^2 x\, dx = \tan x$$

$$u = \sec x \Longrightarrow du = \sec x \tan x\, dx$$

Therefore, we have

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int \sec x \tan^2 x\, dx \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx \\ &= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx\end{aligned}$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx \qquad \text{Collect like integrals}$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Since we developed the integration by parts formula from the Product Rule for derivatives, we would expect many of our examples of this technique to involve a product. However, integration by parts is also useful in cases where the integrand is a quotient, as demonstrated in the next example.

EXAMPLE 7 An integrand involving a quotient

Evaluate

$$\int \frac{xe^x}{(x+1)^2}\, dx$$

Solution: Since $1/(x+1)^2$ is easily integrated, we make the following choices:

$$dv = \frac{dx}{(x+1)^2} \Longrightarrow v = \int \frac{dx}{(x+1)^2} = -\frac{1}{x+1}$$

$$u = xe^x \Longrightarrow du = (xe^x + e^x)\, dx = e^x(x+1)\, dx$$

Thus, we have

$$\begin{aligned}\int \frac{xe^x}{(x+1)^2}\, dx &= xe^x\left(\frac{-1}{x+1}\right) - \int (x+1)e^x\left(\frac{-1}{x+1}\right) dx \\ &= -\frac{xe^x}{x+1} + \int e^x\, dx \\ &= -\frac{xe^x}{x+1} + e^x + C = \frac{e^x}{x+1} + C\end{aligned}$$

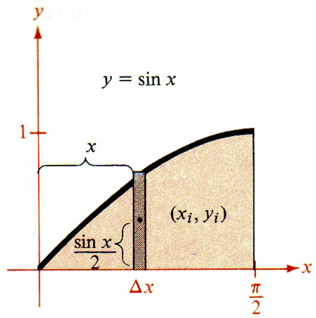

FIGURE 8.2

EXAMPLE 8 An application of integration by parts

Find the centroid of the region bounded by the graph of $y = \sin x$ and the x-axis, $0 \le x \le \pi/2$.

Solution: Using the formulas presented in Section 7.6, together with Figure 8.2, we have

$$\text{area} = A = \int_0^{\pi/2} \sin x\, dx = -\cos x\Big]_0^{\pi/2} = 1$$

$$\bar{y} = \frac{1}{A}\int_0^{\pi/2} \frac{\sin x}{2}(\sin x)\, dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 - \cos 2x)\, dx \qquad \text{Half angle formula}$$

$$= \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{8}$$

$$\bar{x} = \frac{1}{A}\int_0^{\pi/2} x \sin x\, dx$$

Now, using *integration by parts* on this integral, we let $dv = \sin x\, dx$, $u = x$, and obtain $v = -\cos x$ and $du = dx$. Thus, we have

$$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C$$

Now, we determine $\bar{x}$ to be

$$\bar{x} = \Big[-x\cos x + \sin x\Big]_0^{\pi/2} = 1$$

Therefore, we conclude that the centroid of the region is $(1, \pi/8)$. □

As you gain experience in using integration by parts, your skill in determining u and dv will increase. In the following summary, we list several common integrals with suggestions for the choice of u and dv.

SUMMARY OF COMMON INTEGRALS USING INTEGRATION BY PARTS

1. $\int x^n e^{ax}\, dx, \quad \int x^n \sin ax\, dx, \quad \int x^n \cos ax\, dx$

   Let $u = x^n$ and $dv = e^{ax}\, dx$, $\sin ax\, dx$, or $\cos ax\, dx$. (Examples 1, 4)

2. $\int x^n \ln x\, dx, \quad \int x^n \arcsin ax\, dx, \quad \int x^n \arctan ax\, dx$

   Let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and $dv = x^n\, dx$. (Examples 2, 3)

3. $\int e^{ax} \sin bx\, dx, \quad \int e^{ax} \cos bx\, dx$

   Let $u = \sin bx$ or $\cos bx$ and $dv = e^{ax}\, dx$. (Example 5)